DYNAMIC PROBLEMS OF THE THEORY OF CLEAVAGES IN A BEAM APPROXIMATION

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The theory of slender beams is often used to study the equilibrium of cleavages in a narrow strip [1]. In the present work the beam approach is extended to the case of a moving cleavage. Attempts are described in the literature, where the dynamics of cleavages is considered in a beam approximation [2–6]. These are based on replacing the elastic bar, which is a system with an infinite number of degrees of freedom, by a certain other system with one degree of freedom. This, however, is always not permissible.

§1. Derivation of the equations of motion and the boundary conditions. Let us consider the motion of a cleavage along the middle line of an elastic semi-infinite bar ($x \geq 0$) of rectangular cross section with the dimensions given in Fig. 1. The length of the cleavage at the instant t is denoted by $l(t)$. To fix ideas we assume that the faces of the cleavage are loaded at the point $x = 0$ by means of loads of mass m located in a gravitational field with acceleration g. We assume that the bar is weightless and that the material of the bar has the density ρ, Young's modulus E and the density of surface energy T. We shall consider the motion of that half of the bar which lies above the cut. The moment of inertia of this half of the bar about the neutral axis is denoted by $J = 3bH/12$. The deflection of the neutral axis from the non-loaded state is denoted by $u(x, t)$. The transition of the system from one state of motion

$$t = t_1, \quad u = u(x, t_1), \quad \frac{\partial u}{\partial t} = \frac{\partial u(x, t_1)}{\partial t}, \quad l = l(t_1)$$

into another

$$t = t_2, \quad u = u(x, t_2), \quad \frac{\partial u}{\partial t} = \frac{\partial u(x, t_2)}{\partial t}, \quad l = l(t_2)$$

takes place in accordance with the principle of least work in such a way that the extremum of the integral

$$S = \int_{t_1}^{t_2} (K - \Pi)\, dt$$

is realized.

Here K is the kinetic energy and π is the potential energy of the system [7]. We assume that

$$K = \int_0^{l(t)} \frac{\rho bH}{2} \left[\frac{\partial u(x, t)}{\partial t}\right]^2 dx + \frac{m}{2}\left[\frac{\partial u(0, t)}{\partial t}\right]^2. \qquad (1.1)$$

In the expression (1.1) the kinetic energy of the load and the kinetic energy of the vertical displacement of the neutral axis of the beam is taken into account with the assumption that the entire mass of

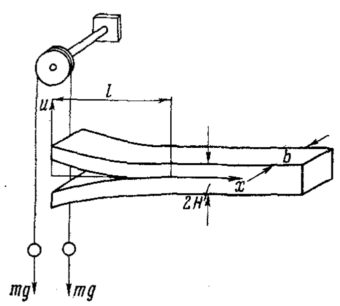

Fig. 1

the beam is concentrated on the neutral axis. The potential energy is taken in the form

$$\Pi = \int_0^{l(t)} \frac{EJ}{2}\left[\frac{\partial^2 u(x, t)}{\partial x^2}\right]^2 dx + mg(R - u(0, t)) + Tbl(t). \qquad (1.2)$$

Here the energy of elastic flexure [8], the potential energy of the load (the constant R is equal to the initial height of the load) and

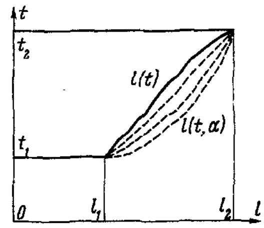

Fig. 2

the surface energy of one face of the cut are taken into account. The expressions (1.1) and (1.2) are approximately true under the conditions

$$H \ll l, \qquad \frac{\partial^2 u}{\partial x^2} \ll \frac{1}{H}, \qquad \frac{\partial^2 u}{\partial x\, \partial t} \ll \frac{1}{H}\frac{\partial u}{\partial t},$$

i. e., when the potential energy of shear can be neglected in comparison with the energy of flexure, and when the direction of the neutral axis and its velocity do not vary significantly over distances which are of the order of the transverse dimension of the bar.

Thus, it is required to find such functions $u(x, t)$ and $l(t)$ which transform the first variation of the functional

$$S = \int_{t_1}^{t_2}\int_0^{l(t)} \left[\frac{\rho bH}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{EJ}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2\right] dx\, dt +$$

$$+ \int_{t_1}^{t_2} \left[\frac{m}{2}\left(\frac{\partial u(0, t)}{\partial t}\right)^2 - mg(R - u(0, t)) - Tbl(t)\right] dt$$

into zero.

As usual, we assume that at the end of the cleavage we have a rigid fixing, i. e., for $x = l(t)$

$$u(l, t) = 0, \quad \partial u(l, t)/\partial x = 0\,. \qquad (1.3)$$

Taking the total derivative with respect to time of the first Eq. (1.3), by virtue of the second Eq. (1.3) we obtain the velocity at the point of fixing

$$\frac{\partial u(l, t)}{\partial t} = 0\,. \qquad (1.4)$$

In Fig. 2 we have shown the plane of the variables x and t, and the region of integration in calculating S: $0 \leq x \leq l(t)$, $t_1 \leq t \leq t_2$. The variation of the functional S is calculated with the condition that both the function $u(x, t)$ and the right hand boundary of the region $l(t)$ are subjected to variation.

To find the unknown functions we proceed using standard variational methods [9]. We introduce families of functions $u(x, t, \alpha)$ and $l(t, \alpha)$ that continuously depend on the parameter α. We assume that these functions are such that all the analytical operations which are necessary in the calculations, are admissible.

In addition, by definition these functions possess the following properties:

$$u(x, t, 0) = u(x, t), \qquad l(t, 0) = l(t), \qquad (1.5)$$

$$u(x, t_1, \alpha) = u(x, t_1), \qquad u(x, t_2, \alpha) = u(x, t_2), \qquad (1.6)$$

$$u\,[l\,(t,\,\alpha),\,t,\,\alpha]=0,\qquad \frac{\partial u\,[l\,(t,\,\alpha),\,t,\,\alpha]}{\partial x}=0 \tag{1.7}$$

The functional S is considered as a function of α. We introduce the notations

$$\delta u\,(x,\,t)=\frac{\partial u\,(x,\,t,\,0)}{\partial\alpha}\,\alpha,\quad \delta l\,(t)=\frac{\partial l\,(t,\,0)}{\partial\alpha}\,\alpha,\quad \delta S=\frac{\partial S\,(0)}{\partial\alpha}\,\alpha\,. \tag{1.8}$$

From the definitions (1.8) and the conditions (1.6) it follows that

$$\delta u\,(x\,,\,t_1)=\delta u\,(x,\,t_2)=0\,. \tag{1.9}$$

Furthermore

$$\delta u\,[l\,(t),\,t]=0,\qquad \frac{\partial\delta u\,[l\,(t),\,t]}{\partial x}=-\frac{\partial^2 u\,[l\,(t),\,t]}{\partial x^2}\,\delta l\,. \tag{1.10}$$

To obtain the conditions (1.10), it is sufficient to represent the quantity $u[l(t), t, \alpha]$ in a Taylor series of powers of the difference $l(t) - l(t, \alpha)$

$$u\,[l\,(t),\,t,\,\alpha]=\sum_{n=0}^{\infty}\frac{1}{n!}\,\frac{\partial^n u\,[l\,(t,\,\alpha),\,t,\,\alpha]}{\partial x^n}\,[l\,(t)-l\,(t,\,\alpha)]^n$$

and to use the definition (1.8) and the properties of Eqs. (1.5), (1.7).

Let us calculate δS. We first calculate the variation $\delta_1 S$, assuming that the boundary $l(t)$ is fixed

$$\delta_1 S=\int_{t_1}^{t_2}\int_{0}^{l(t)}\left[\rho bH\,\frac{\partial u}{\partial t}\,\frac{\partial\delta u}{\partial t}-EJ\,\frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2\delta u}{\partial x^2}\right]dx\,dt+$$

$$+\int_{t_1}^{t_2}\left[m\,\frac{\partial u\,(0,\,t)}{\partial t}\,\frac{\partial\delta u\,(0,\,t)}{\partial t}+mg\delta u\,(0,\,t)\right]dt\,.$$

We transform the double integral, applying the formula of Green

$$\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial N}{\partial t}\right)dx\,dt=\int N\,dx+Q\,dt\;,$$

where the integral on the right side of this formula is taken along the boundary of the region of integration in the double integral. We go counter clockwise, if the x, t coordinate system is a right-handed one;

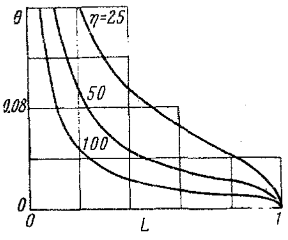

Fig. 3

in a contrary case we go clockwise. The first term in the single integral is transformed by integrating by parts.

Taking into account the condition (1.4) and the boundary values of the variations (1.9), (1.10), we obtain

$$\delta_1 S=\int_{t_1}^{t_2}\left\{EJ\left[\frac{\partial^2 u\,(l,\,t)}{\partial x^2}\right]^2\delta l+EJ\,\frac{\partial^2 u\,(0,\,t)}{\partial x^2}\,\frac{\partial\delta u\,(0,\,t)}{\partial x}-\right.$$

$$-EJ\,\frac{\partial^3 u\,(0,\,t)}{\partial x^3}\,\delta u\,(0,t)+$$

$$\left.+\,m\left[g-\frac{\partial^2 u\,(0,\,t)}{\partial t^2}\right]\delta u(0,\,t)\right\}dt+$$

$$+\int_{t_1}^{t_2}\int_{0}^{l(t)}\left[EJ\,\frac{\partial^4 u}{\partial x^4}+\rho bH\,\frac{\partial^2 u}{\partial t^2}\right]\delta u\,dx\,dt\,.$$

We calculate the part $\delta_2 S$ of the variation which arises in varying the boundary $l(t)$

$$\delta_2 S=\int_{t_1}^{t_2}\left\{-\frac{EJ}{2}\left[\frac{\partial^2 u\,(l,\,t)}{\partial x^2}\right]^2-Tb\right\}\delta l\,dt\,.$$

In obtaining $\delta_2 S$ we have used Eq. (1.4) and the rule of differentiating a definite integral with respect to the upper limit.

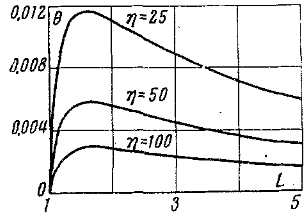

Fig. 4

Stipulating that $\delta S = \delta_1 S + \delta_2 S = 0$, we obtain the equation of motion

$$\frac{\partial^4 u\,(x,\,t)}{\partial x^4}+\frac{1}{a^2}\,\frac{\partial^2 u\,(x,\,t)}{\partial t^2}=0,\qquad a^2=\frac{EJ}{\rho bH} \tag{1.11}$$

and the natural boundary conditions

$$\frac{\partial^2 u}{\partial x^2}=0,\qquad EJ\,\frac{\partial^3 u}{\partial x^3}=m\left(g-\frac{\partial^2 u}{\partial t^2}\right)\qquad (x=0)\;, \tag{1.12}$$

$$\frac{\partial^2 u\,(l,\,t)}{\partial x^2}=A,\qquad A=\left(\frac{2Tb}{EJ}\right)^{1/2}. \tag{1.13}$$

Furthermore, the conditions (1.3) are satisfied for $x = l(t)$. To determine completely the motion of the system under consideration, we must specify, at a certain initial instant, a distribution of the displacements and velocities that is compatible with the boundary conditions.

Equations (1.11), (1.12) and (1.13) together with the corresponding initial conditions describe the motion of a beam with one end rigidly fixed at $x = l(t)$, and with a zero moment and shear force due to the weight of the mass specified at the end $x = 0$. The conditions at the left end depend on the method of loading and can be specified immediately, proceeding from the fact that the moment and shear force are proportional respectively to the second and third derivative of the displacement with respect to the coordinate. It is easy to show that in the case where distributed normal loads are applied to the faces of the cleavage, Eq. (1.11) is transformed into a nonhomogeneous equation. Here a function equal to the linear density of the load divided by the flexural rigidity stands on its right side.

The conditions (1.11), (1.12) and (1.3) enable the solution to be found for any law of motion $l(t)$ of the point of fixing. As follows from the conclusion, in reality only such motions are realized for which the condition (1.13) is fulfilled. Consequently, this condition, within the limits of applicability of the theory of thin beams to the study of cleavages, plays the same part as the condition of finite stresses of G. I. Barenblatt [1] does in the general case elastic-brittle failure. We note that in [2-6], in considering the motion of a cleavage, the static solution for a clamped beam of length l, loaded arbitrarily at the free end, was taken as the solution $u(x, t)$. At the same time it was assumed that the parameter l depends on time. As a result, the conditions at the loaded end and the conditions of fixing were found to be fulfilled, but Eq. (1.11) and the condition (1.13) were violated almost everywhere (with the exception of the position of equilibrium). The further away from the position of equilibrium was the system, the stronger was this violation.

§2. **Examples.** a) Stationary motion. Let us consider a solution of the form $u(x, t) = u(x - Vt)$, where V is a constant having the dimensions of velocity, for the boundary conditions specified on the lines $\chi = x - Vt = \text{const}$ and depending only on χ.

In the case of stationary wedging by a wedge of thickness 2h the boundary conditions are specified as follows:

$$u = h, \qquad \partial^2 u/\partial x^2 = 0, \qquad (x = Vt),$$
$$u = 0, \qquad \partial u/\partial x = 0, \qquad \partial^2 u/\partial x^2 = A \quad (x = Vt + l). \quad (2.1)$$

Under these conditions l is the unknown constant length of the free cleavage in front of the wedge.

If in Eq. (1.1) and in the boundary conditions (2.1) we transform into χ, then we obtain an ordinary differential equation and boundary conditions for $\chi = 0$, $\chi = l$ for the determination of the function $u(\chi)$. Using the conditions (2.1), we obtain the displacement

$$u(\chi) = h\left[\left(-\frac{a^2}{V^2}\sin\frac{V\chi}{a} + \chi\frac{a}{V}\cos\frac{Vl}{a}\right) \times \left(\frac{a^2}{V^2}\sin\frac{Vl}{a} - l\frac{a}{V}\cos\frac{Vl}{a}\right)^{-1} + 1\right]. \quad (2.2)$$

It is necessary to point out that Eq. (2.2) defines the displacement only in the region $0 \le \chi \le l$. When $\chi < 0$, the solution is sought such that $u(0)$ and $u'(0)$ would be continuous, with $u''(-0)$ and $u'''(-0)$ being 0. This ensures the necessary jump of the third derivative in passing through the point of application of the concentrated force.

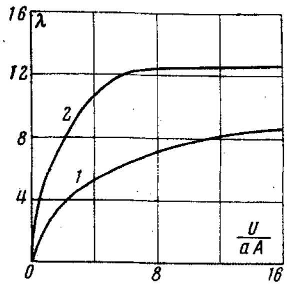

Fig. 5

From the last condition (2.1) we obtain the equation for determining l:

$$h\sin\frac{Vl}{a}\left(\frac{a^2}{V^2}\sin\frac{Vl}{a} - \frac{a}{V}l\cos\frac{Vl}{a}\right)^{-1} = A. \quad (2.3)$$

Going to the limit $V \to 0$ in Eq. (2.3), we obtain the equilibrium length l_0 of the cleavage before the stationary wedge:

$$l_0^4 = \frac{9h^2}{A^2} = \frac{3EH^3h^2}{8T}. \quad (2.4)$$

We rewrite Eq. (2.4) in the dimensionless form

$$\frac{1}{3L^2} = \frac{1}{12\eta^2\theta^2L^2} - \frac{1}{\sqrt{12}\,\eta\theta L}\operatorname{ctg}\sqrt{12}\,\eta\theta L \quad \left(\theta^2 = \frac{V^2 p}{E},\right.$$
$$\left. L = \frac{l}{l_0}, \quad \eta = \frac{l_0}{H}\right). \quad (2.5)$$

The relation (2.5) is shown in Fig. 3 in the L, θ coordinates for various values of the parameter η. When the cleaving velocity V increases to infinity, the length as 1/V tends to 0. The absence of any special features in passing through the velocity of propagation of the surface waves is due to the approximation (1.1), (1.2) adopted here. The absence of a limiting velocity of the flexural waves (their velocity increases as the length shortens) is characteristic for this approximation [8]. Thus, for small 1/H the beam approach is not applicable, and a more accurate analysis is required.

Let us consider the stationary motion of a cleavage when concentrated forces p are applied at a distance l from its end. The boundary conditions for this case will be

$$\frac{\partial^2 u}{\partial x^2} = 0, \qquad EJ\frac{\partial^3 u}{\partial x^3} = p \qquad (x = Vt),$$
$$u = 0, \qquad \frac{\partial u}{\partial x} = 0, \qquad \frac{\partial^2 u}{\partial x^2} = A \qquad (x = Vt + l). \quad (2.6)$$

Proceeding as before, we obtain the connection between the velocity of motion V and the length l:

$$\frac{a}{V}\sin\frac{Vl}{a} = \frac{AEJ}{p}. \quad (2.7)$$

Going to the limit $V \to 0$, we obtain the equilibrium length of the cleavage:

$$l_0 = \frac{AEJ}{p} = \frac{\sqrt{Eb^2H^3T}}{\sqrt{6}\,p}. \quad (2.8)$$

It is well known that the equilibrium in this case is unstable. Using the notations (2.6), but determining l_0 from Eq. (2.8), we rewrite Eq. (2.7) in the dimensionless form

$$\sqrt{12}\,\theta\eta = \sin\sqrt{12}\,\theta\eta L. \quad (2.9)$$

As is seen from Eq. (2.9), a stationary motion is possible only for $L \ge 1$, $\sqrt{12}\,\theta\eta \leqslant 1$. Any velocity of motion $V \ne 0$ corresponds to infinitely many lengths of the cleavage. Considering only positive values of the sine, which corresponds to a motion in the positive direction of the abscissa axis, we can write

$$2\pi n \leqslant \sqrt{12}\,\theta\eta L \leqslant \pi(2n + 1) \qquad (n = 0, 1, 2, 3, \ldots).$$

Let at the beginning n = 0. The relation (2.9) for this case is given in Fig. 4 in the L, θ coordinates for various values of η. Two states of motion correspond to each velocity. When $1 \leqslant L \leqslant {}^1/_2\,\pi$, an increase in l, for the unaltered velocity V and the force p, must cause an increase in A to preserve the equality (2.7). It is intuitively clear that, since A is constant, the cleavage abandons the stationary state and begins to accelerate; this causes a further increase in the length l. Exactly in the same way a random shortening of the cleavage leads to it being shut with a snap. The stationary motion for $1 \leqslant L \leqslant {}^1/_2\,\pi$ is consequently unstable. From similar considerations it follows that for $L > {}^1/_2\,\pi$ the necessary condition of stability of the stationary state is fulfilled: for a random increase in l the cleavage tends to reduce its velocity, whereas for a reduction it tends to increase its velocity.

In the case of a natural number n the length l differs from the length given in Fig. 4 by the amount $(2\pi H/\sqrt{12}\,\theta)\,n$. The factor of n represents the length of the sinusoidal flexure wave, which is capable of propagating along the bar with the velocity V. Two states of motion are possible for each n: if $2\pi n \leqslant \sqrt{12}\,\theta\eta L \leqslant \pi(2n + {}^1/_2)$, then the motion is unstable; if, however, $\pi(2n + {}^1/_2) < \sqrt{12}\,\theta\eta L < \pi(2n + 1)$, then the necessary stability conditions mentioned above are satisfied. The behavior of the system after it leaves the stationary state remains unclear, since then already nonstationary motions must be considered for its study. We can only point out the a priori possibility of the system to pass from the unstable state into a stable on (if the latter exists). Precisely which n is realized in the reality, depends on which displacements and velocities are specified at the initial instant to secure the subsequent stationary motion.

It is not difficult to analyze the case where the load is distributed over the faces of the cleavage, and in a steady state moves with it.

b) The motion of a cleavage when its faces at the origin of the coordinates are moved apart with a constant velocity U. Let us consider the motion of a cleavage in a semi-infinite bar for the following boundary conditions:

$$u(0, t) = Ut, \qquad \frac{\partial^2 u(0, t)}{\partial x^2} = 0,$$
$$u(l, t) = 0, \qquad \frac{\partial u(l, t)}{\partial x} = 0, \qquad \frac{\partial^2 u(l, t)}{\partial x^2} = A. \quad (2.10)$$

Equation (1.11) and the condition (2.10) are invariant with respect to the group of transformations of the scales $u = k^2 u_1$, $x = kx_1$, $t = k^2 t_1$, where k is any positive number. Therefore the solution is sought for as a relation between the invariants of this group:

$$\frac{u(x, t)}{Ut} = f(\xi), \qquad \xi = \frac{x^2}{at}. \quad (2.11)$$

Substituting Eqs. (2.11) into Eq. (1.11) and the conditions (2.10), we obtain the following equation for determining the function $f(\xi)$:

$$\frac{d^4f}{d\xi^4}+\frac{3}{\xi}\frac{d^3f}{d\xi^3}+\left(\frac{3}{4\xi^2}+\frac{1}{16}\right)\frac{d^2f}{d\xi^2}=0 \qquad (2.12)$$

and the boundary conditions

$$f(0)=1, \quad \lim_{\xi\to 0}\,[4\xi f''(\xi)+2f'(\xi)]=0, \qquad (2.13)$$

$$f(\lambda)=0, \qquad f'(\lambda)=0,$$

$$4\lambda f''(\lambda)=\frac{aA}{U}, \qquad \lambda=\frac{l^2}{at}\,. \qquad (2.14)$$

Using the substitution $d^2f/d\xi^2 = F(\xi)\xi^{-3/2}$ we reduce Eq. (2.12) into a linear equation with constant coefficients. Its solution yields

$$\frac{d^2f}{d\xi^2}=\xi^{-3/2}\left(D\sin\frac{\xi}{4}+E\cos\frac{\xi}{4}\right)=$$

$$=D\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n-1/2}}{4^{2n+1}(2n+1)!}+E\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n-3/2}}{4^{2n}(2n)!},$$

where D and E are arbitrary constants. Integrating twice, we obtain the general solution of Eq. (2.12)

$$f(\xi)=D\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+3/2}}{4^{2n+1}(2n+1)!\,(2n+1/2)\,(2n+3/2)}+$$

$$+E\sum_{n=0}^{\infty}\frac{(-1)^n\xi^{2n+1/2}}{4^{2n}(2n)!\,(2n-1/2)\,(2n+1/2)}+B\xi+C\,, \qquad (2.15)$$

where B and C are constants of integration.

From the conditions (2.13) it follows that B = 0, C = 1. D and E are determined from the first and second conditions (2.14) as functions of λ. The third condition (2.14) enables λ to be found as a function of the velocity U of the faces moving apart. Thus

$$l(t)=\sqrt{\lambda(U)/aA)at}\,. \qquad (2.16)$$

The velocity of the cleavage varies inversely proportionally to its length:

$$\frac{dl}{dt}=\frac{a\lambda\,(U/aA)}{2l}. \qquad (2.17)$$

The graph of the function λ is given in Fig. 5 (curve 1). The boundary condition $\partial^2u(0,t)/\partial x^2=0$ allows us to generalize: for x = 0 we can specify any bending moment that is constant with time.

Analogously we can investigate a motion that is symmetrical about the point x = 0 in an infinite bar $(-\infty<x<\infty)$. The difference consists only of the fact that instead of the bending moment specified at x = 0 we must impose the condition $\partial u(0,t)/\partial x=0$. The half length l of the cleavage again has the form (2.15), the function λ is different. Its graph is shown in Fig. 5 (curve 2). The shear force at the origin of the coordinates in both cases varies inversely proportionally to the length of the cleavage. The occurrence of arbitrarily large velocities of motion of the cleavage for small lengths is explained exactly in the same way as in the case of stationary wedging. The meaning of the solution thus obtained is not that it can be used to describe the motion of short cleavages. It means that if at any instant of time the distribution of the displacements and velocities is described by the formulas (2.11), (2.15), then the subsequent motion will be a self-modeling motion described by the same formulas. If, however, the initial data has an arbitrary form $u(x',0)$, $\partial u(x',0)/\partial t$, then from the dimensional analysis we have

$$u(x,t)=Ut\varphi\left(\frac{x^2}{at},\ \frac{u(x',0)}{Ut},\ \frac{1}{U}\frac{\partial u(x',0)}{\partial t},\ \frac{aA}{U}\right)$$

instead of the expression (2.11), where φ is a dimensionless function of its arguments. As the time increases, the dependence on time is retained only in the first argument. The solution for $t\to\infty$ tends to the form (2.11), and the velocity of the cleavage tends to the form (2.17). At the same time the function λ is determined not only by the quantity U/aA but also by the initial data.

The examples considered here allow us to make the following statement: the inertia term in Eq. (1.11) must be considered when the ratio of the velocity of the cleavage to the velocity of the flexural waves, having the same length as the cleavage, cannot be neglected in comparison with unity.

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